Pythagorese Dissections

by Edo Timmermans

In December 1989, the leaflet Supremum supplied by the mathematics and computer science faculty at the Technical University of Eindhoven contained a problem that triggered my fascination for dissections. The question was to cut a 13×13-square into 4 parts, with which one can also form both a 5×5-square and a 12×12-square.
With a computer program I found solutions for both $5^2+12^2=13^2$ and $8^2+15^2=17^2$. After observing these solutions carefully, I noticed that they could be used to find several infinite sets of related solutions. These sets are based on two infinite series of solutions with whole numbers of the equation $a^2+b^2=c^2$. I only looked for dissections where none of the parts need to be mirrored.
The first kind of series a, b and c can be found with $a=2n+3$, $b=2n^2+6n+4$ and $c=b+1$. For the value of n, take $n\geq 0$. With increasing n, this leads to the equations $3^2+4^2=5^2$, $5^2+12^2=13^2$, $7^2+24^2=25^2$, $9^2+40^2=41^2$, $11^2+60^2=61^2$, etcetera.
The second kind can be found with $a=2k+2$, $b=k^2+2k$ and $c=b+2$. Take variable k to be a positive odd number. With increasing k, this leads to the equations $4^2+3^2=5^2$, $8^2+15^2=17^2$, $12^2+35^2=37^2$, $16^2+63^2=65^2$, $20^2+99^2=101^2$, etcetera.
Amazingly, there are series where the number of solutions per equation increases as the squared numbers enlarge. Drawing formulas for such cases contain variable p.
Not every dissection can be used as a basis for an infinite series. For each of the equations $5^2+12^2=13^2$ and $8^2+15^2=17^2$, such a dissection exists; see figure 1.

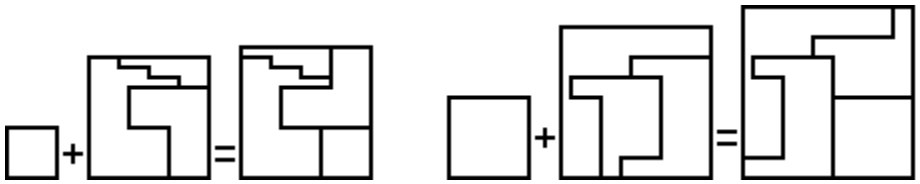

Figure 1. Two in-expandable dissections; $5^2+12^2=13^2$ (left) and $8^2+15^2=17^2$ (right)

Description for using the drawing formulas

In order to make drawings for these infinite sets of shapes, I created drawing formulas. Generally, start by drawing an a×a-square and a b×b-square. Within the b×b-square the drawing takes place. Once the drawing is finished, this square is divided in 3 parts. Together with the a×a-square, they can form the c×c-square.
The left-down coordinates from the b-square are (0,0); the up-right coordinates are (b,b). Every drawing formula begins with start coordinates, followed by a number of drawing actions. Keep the pencil on the paper during these actions; only lift it when new start coordinates are given.
Capital letters indicate the drawing directions. L (left) decreases the first coordinate; R (right) increases it. U (up) increases the second coordinate; D (down) decreases it.
Such a capital letter is always followed by a number or a function that gives a number when the variables are filled in. This number indicates the amount of units to move. A unit is a partial straight line, horizontal or vertical, of length 1. Occasionally, units may be drawn twice. Keeping the formulas as short as possible causes that.
As an example, observe the end partial formula UaLan-nU1Ra-1. When n=1 then a=5. In that case, move 5 units up, then 5*1-1 (=4) units left, then 1 unit up and finally 4 units right. When n=0 nothing happens during the action Lan-n.

Straight brackets stand for a repetition. The number directly following '[' indicates the amount of times the following drawing actions must be performed. For example, [n/2Da-1La+1] means that the action 'go a-1 units down, then go a+1 units left' must be repeated n/2 times; when the result of a division is not an integer, it must be rounded down. Variables a, b and c can be calculated after choosing n or k, using the formulas given earlier.
In formulas C and D variable e occurs, due to an alternating feature within these series. When n is even then e=0, otherwise e=1.
In most drawing formulas starting with a capital letter, one can choose n≥0. However, when the variable p occurs, then one must choose n≥p after choosing p≥0. Exceptions are formulas O, P and W; there one must choose p≥2. In formula J, additionally, p can have value n+1, as long as n≥0. In formulas K, L, O, P, S and W variable p must be even, but odd in M, N, R, T, U, and V. There is a peculiar alternating relationship between formulas K..P and Q..W, see figure 3.
In the remaining formulas a..f, variable k must be chosen. Generally, one must choose k≥3, where k must be odd as well. Variable k can have value 1 in formula a, when accepting that L-1=R1 and R-1=L1. When variable p occurs, choose 1≤p≤k/2. Also, in formulas c and d variable p must be odd, whereas even in e and f. Because of this, variable k cannot have value 3 in formulas e and f.

The drawing formulas

A: (c-a,0)Ua+1[nLa-1Ua+1](b,b-a)Lc-a
B: (c-a,0)UaR1U1[nLaUaR1U1](b,b-a)Lc-a
C: (0,c-a)[nRa+1Da-1]Ra+1(b-a,a-1)Ub-a[n/2La-1Da+1]Ln-en-e+1Lea-eDen+2e
D: (a-1,b)[nDa+1Ra-1]Da+1(b,b-a)L1[n/2Da-1La+1]Dn-en-e+1Dea-eLen+2e
E: (0,1)Rb-1Ua-2R1L1[nLa-1D2]La-1D1

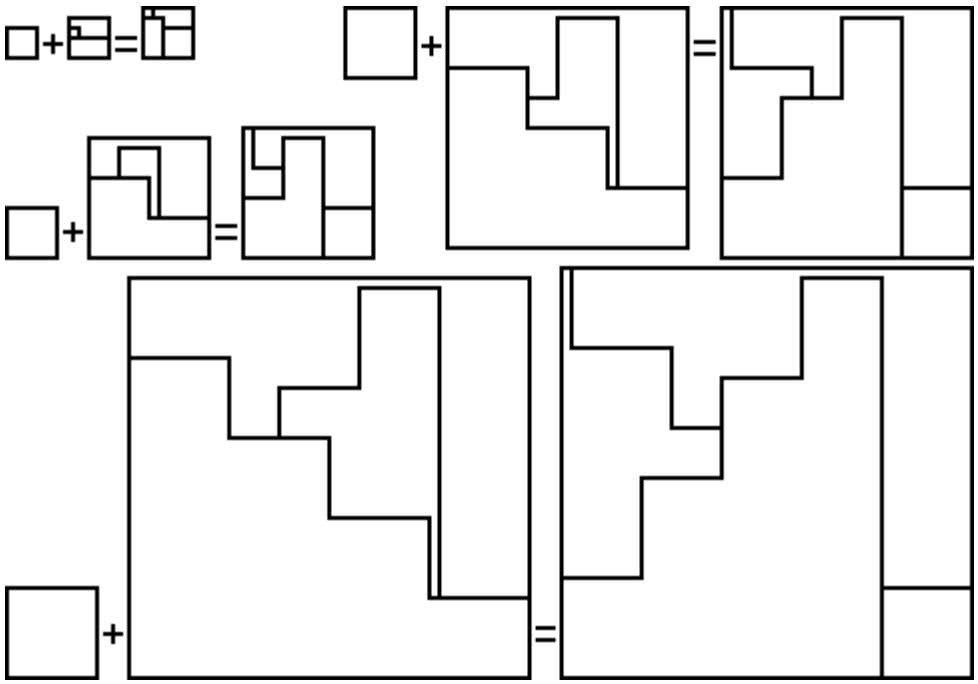

Figure 2. Choose n=0, 1, 2 and 3 in formula C, with e=0, 1, 0 and 1

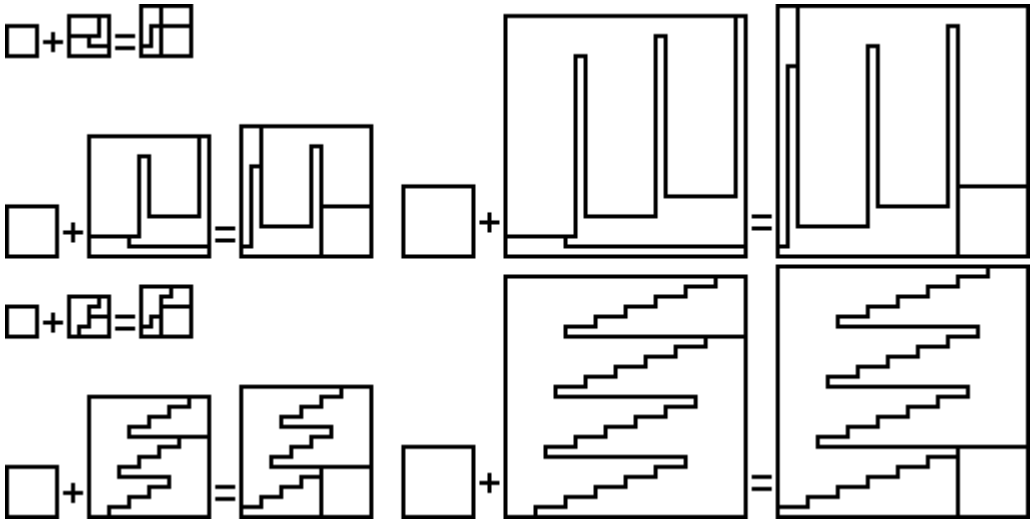

Figure 3. Choose n=0, n=1 and n=2 in formula G; choose n=0, n=1 and n=2 combined with p=n+1 in formula J

F: (a-1,0)[nU1Ra-1]U1Ra-2UaLan-n[nU1Ra-1]R1(b-a,n)UaLan-nU1Ra-1
G: (0,2)[nRaUc-aR1Dc-a-2]RaUc-a(b,1)Lc-aU1
H: (1,b)[b-aD1R1]Da[n+1Ua-1R2]
I: (c-a,0)[n+1Ua-1R2]L1Dc-a
J: (n+1,0)[pU1[a-2Rn+1U1]Lan][a-1U1Rn+1](b,c-a)[n+1-pL1Da-1]Ln+2
K: (0,1)Ra-2U1[p/2Ra-1UaL2U1][n-pRa-1U2][p/2RaDa-1R1U2]Ra
(0,b-a+2)R1Da-1[p/2RaU2R1Da-1][n-pR2Da-1]
L: (0,1)Ra-2[p/2U1Ra-1UaL2]Ua-1[n-pL2Ua-1]Ra-2[n-pU2Ra-1]
Da-2[n-pR2Da-1]R2[p/2Da-1R1U2Ra](0,b-a+2)R1[p/2Da-1RaU2R1]
M: (0,1)Ra-2U1[p/2Ra-1UaL2U1][n-p+1Ra-1U2][p/2RaDa-1R1U2]Ra
(0,b-a+2)R1Da-1[p/2RaU2R1Da-1]Ra[n-pU2Ra-1]Da-2[n-pR2Da-1]
N: (0,1)Ra-2[p/2U1Ra-1UaL2][n-p+1U1Ra-2Ua-2La]U1[n-p+1Ra-2UaR1Da-2]
[n-p+1RaD1La-2Da-2]Ra[p/2Da-1R1U2Ra](0,b-a+2)R1[p/2Da-1RaU2R1]Da-2
O: (0,1)Ra-2U1Ra-1[p/2-1UaL2U1Ra-1][n-p+1UaRa-2Da-2R1]U2Ra
[p/2-1Da-1R1U2Ra]
(0,b-a+2)R1[p/2-1Da-1RaU2R1]Da-1RaDa-2[n-p+1La-2D1RaDa-2]
P: (0,1)Ra-2U1Ra-1[p/2-1UaL2U1Ra-1][n-p+1UaRa-2Da-2R1]U2Ra
[p/2-1Da-1R1U2Ra]
(0,b-a+2)R1[p/2Da-1RaU2R1][n-pDa-2Ra-2UaR1][n-p+1Da-2RaD1La-2]
Q: (0,a-2)Ra-1[nD2Ra-1](c-a,0)[n+1Ua-1R2]
R: (0,a-2)[p/2R1Ua-1RaD2]R1Ua-1Ra[n-pD2Ra-1](c-a,0)
[p/2+1U1La-1UaR2][n-pUa-1R2][p/2+1Ua-1R1D2Ra]
S: (0,a-2)[p/2R1Ua-1RaD2][n-pRa-1D2][p/2Ra-1D1L2Da]Ra-1D1
(b,c-a)[p/2LaU2L1Da-1]La[n-pU2La-1]Da-2[n-pL2Da-1]
T: (0,a-2)[p/2R1Ua-1RaD2]R1[n-p+1Ua-1R2]Ua-2[n-p+1Ra-1D2]Ra
[p/2Ua-1R1D2Ra](c-a,0)[p/2+1U1La-1UaR2]Ua-1[n-pR2Ua-1]
U: (0,a-2)[p/2R1Ua-1RaD2][n-p+1R1Ua-2Ra-2Da]R1(b,c-a)
[p/2LaU2L1Da-1][n-p+2LaDa-2Ra-2D1][p/2L2DaRa-1D1]
V: (0,a-2)[p/2R1Ua-1RaD2][n-p+1R1Ua-2Ra-2Da][p/2Ra-1D1L2Da]Ra-1D1
(b,c-a)[p/2LaU2L1Da-1]LaU2L1Da-2[n-pLa-2UaL1Da-2]LaD1
[n-pRa-2Da-2LaD1]
W: (0,a-2)[p/2-1R1Ua-1RaD2]R1Ua-1Ra[n-p+1Ua-2La-2U1Ra][n-p+1Ua-2R1Da
Ra-2][n-p+1D1LaDa-2Ra-2]D1[p/2L2DaRa-1D1](b,c-a)[p/2LaU2L1Da-1]

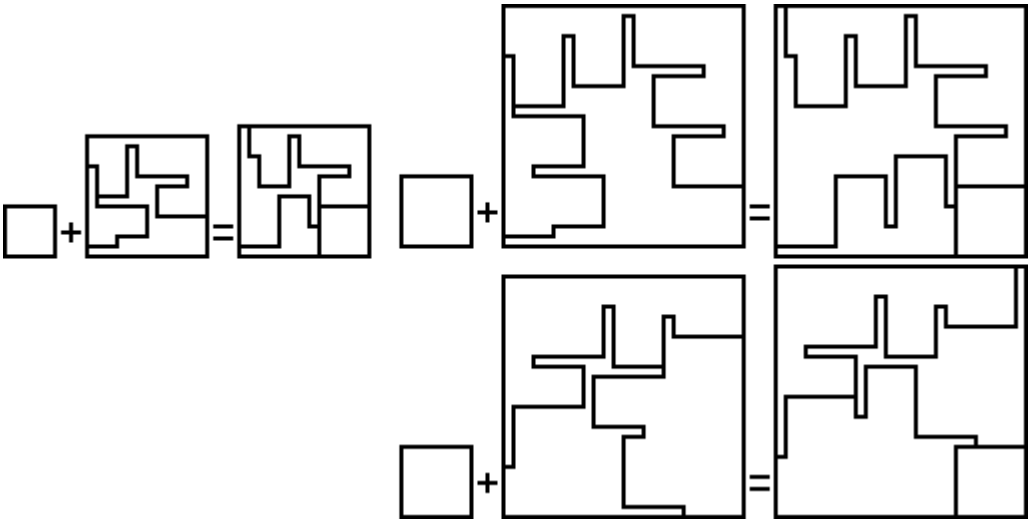

Figure 4. Choose n=1 and n=2 in formula N and n=2 in formula W

a: (2k,0)U2[k/2R2kU2]Lb-4k[k/2U2R2k](3k,k+1)L2kD2Rb-4k
b: (2k,0)[k/2U2R2k]UaLb-4k(b-3k,k-1)U2kLb-4k[k/2+1U2R2k]
c: (2k,0)U2[k/2R2kU2]Lb-4k[p/2+1UaRa-4]R2[k/2-pU2R2k][p/2Da-4Ra]
(b-3k,k-1)Lb-4k[p/2+1UaRa-4]R2
d: (2k,0)U2[k/2R2kU2]Lb-4k[p/2+1UaRa-4][k/2-pU2R2k]R2[p/2Da-4Ra]
(b-3k,k-1)Lb-4k[p/2+1UaRa-4]U2R2
e: (2k,0)U2[k/2R2kU2]Lb-4k[p/2UaRa-4][k/2-p+1U2R2k]R2[p/2-1Da-4Ra]
(b-3k,k-1)Lb-4k[p/2UaRa-4]Ua[k/2-pL2U2k]R2k[k/2-pU2R2k]
[k/2-pD2kR2]Da-4
f: (2k,0)U2[k/2R2kU2]Lb-4k[p/2UaRa-4]R2[k/2-p+1U2R2k][p/2-1Da-4Ra]
(b-3k,k-1)Lb-4k[p/2UaRa-4][k/2-p+1U2kL2][k/2-p+1U2R2k]U2R2
[k/2-p+1D2kR2]

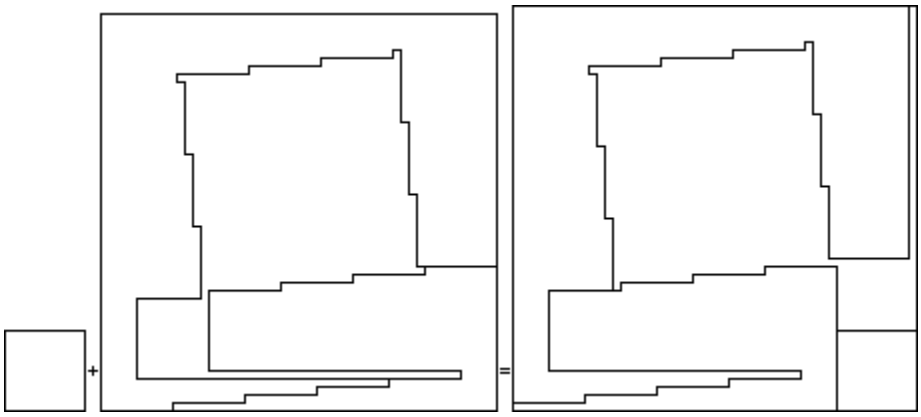

Figure 5. Choose n=9 and p=2 in formula f, with a=20, b=99 and c=101

For more drawings, please have a look at the website, cff.helm.lu/NKC.html.

References

[1] puzzelhoek, Supremum, jaargang 22 nr. 2 (1989), Technical Univ. of Eindhoven
[2] Edo Timmermans, Pythagoreïsche dissecties, Pythagoras, Jan. 2005, pp 28-32.