Solving *π/6 Wrecked Angles*
by John Blasdale

Introduction

Ben Tanoff's *π/6 Wrecked Angles* [1] inspired me to write up the following solution. My own particular interest in puzzles consists in developing proof-like solutions to various puzzles of interest — mainly put-together puzzles. I usually avoid computer-based search techniques because they often reveal little of interest about the puzzle structure or fail to provide anywhere near the same pleasure as applying logic, common sense and a bit of mathematics. I was able to solve *π/6 Wrecked Angles* by focusing on the assignment of the triangle edges to the rectangle boundary. Interestingly, most of the solution takes place without the need to visualise the puzzle. I have not yet created a set of puzzle pieces and shuffled them about. Since the puzzle is quite modest, this might be a much faster way to arrive at a solution! However, it always leaves uncertain the question of whether there may be more than one solution. I hope I have addressed this fully below.

Puzzle Components

The puzzle comprises the six triangles pictured in Figure 1. The edge lengths of the triangles are listed in Table 1 (see Ben's article [1] for a larger illustration with included dimensions). Note that all triangle edges with a √3 denominator in the original CFF-article have been normalised in order to bring all √3's into the numerator, see Table 1 (using the equality 1/√3 = √3/3). The total area of the six triangles is 4√3, and the goal is to fit the six pieces into a rectangle of that area.

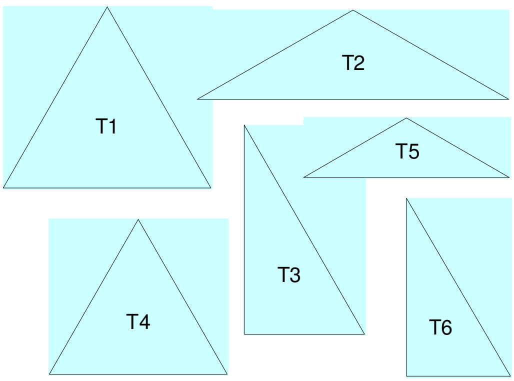

Figure 1. The six puzzle pieces

Triangle	Edge 1	Edge 2	Edge 3	Right Angle
T1	2	2	2	No
T2	3	√3	√3	No
T3	2	4/3 √3	2/3 √3	Yes
T4	√3	√3	√3	No
T5	2	2/3 √3	2/3 √3	No
T6	1	2	√3	Yes

Table 1. Lengths of the six triangle edges

Rectangle Dimensions

First we will apply a bit of algebra to work out the dimensions of the rectangle into which the pieces fit. Label the dimensions of the rectangle a and b so that $ab = 4\sqrt{3}$. Since all triangle edges are either integers or rational multiples of √3, we know the rectangle dimensions can only be combinations of such numbers.

So suppose $a = a_1 + a_2\sqrt{3}$ and $b = b_1 + b_2\sqrt{3}$ (1)

where a_1, b_1 are both integer and a_2, b_2 are both rational

then $(a_1 + a_2\sqrt{3})(b_1 + b_2\sqrt{3}) = 4\sqrt{3}$

and so $a_1b_1 + 3a_2b_2 + (a_1b_2 + a_2b_1)\sqrt{3} = 4\sqrt{3}$

Now an arithmetic function of rational numbers cannot yield an irrational result,

implying that $a_1b_1 + 3a_2b_2 = 0$ (2)
and thus $(a_1b_2 + a_2b_1) = 4$ (3)

From (3) it follows that a_1 and a_2 cannot both be zero; likewise b_1 and b_2 cannot both be zero. So, in order for the expression in (2) to be zero,

either $a_1 = 0$ and $b_2 = 0$
or $b_1 = 0$ and $a_2 = 0$

Substituting in (1) $a = a_2\sqrt{3}$ and $b = b_1$ (4)
or $a = a_1$ and $b = b_2\sqrt{3}$ (5)

Equations (4) and (5) are fully equivalent so, without loss of generality, we will use equation (5) and assume that *a* is an integer and *b* is a rational multiple of √3. So the potential values of *a* and *b* at this stage are shown in Table 2 (The comments derive from the discussion that follows).

a	***b***	**Comment**
1	4√3	Not possible. Side *a* is too narrow.
2	2√3	
3	4/3 √3	Not possible. See following discussion.
4	√3	
5	4/5 √3	Not possible. Side *b* is too narrow.
etc.	etc.	Not possible. Side *b* is too narrow.

Table 2. Possible rectangle dimensions *a* and *b*

When side *a* is sufficiently small (< 2) or sufficiently large (> 4), the rectangle is too narrow to accommodate triangle T1, whose edges are 2, 2, 2 and thus has a minimum width (a perpendicular from any vertex to the opposite edge) of √3. This leaves only the cases where *a* = 2, 3 or 4. We will shortly eliminate the case *a* = 3 too, but we first need to think some more about triangle edges and then apply some sneaky reasoning to triangles T3 and T5.

Triangle edge length parity
Triangle edges lie either on the sides of the rectangle or within the rectangle interior. Since the rectangle has equal and opposite sides the total length of triangle edges for a pair of opposite rectangle sides must be divisible by 2.

Triangle edges in the rectangle interior must lie along interior lines. The two sides of each such line must be balanced in terms of the total integer contribution and the total √3 multiple. Even if an interior line had a length of the form $u = u_1 + u_2\sqrt{3}$ and was opposed by a line of equal length $v = v_1 + v_2\sqrt{3}$, so that $u_1 + u_2\sqrt{3} = v_1 + v_2\sqrt{3}$ then $u_1 - v_1 = (v_2 - u_2)\sqrt{3}$. This can only be true if $u_1 = v_1$ and $v_2 = u_2$, which means that integer lengths require equal, opposing lengths; and likewise for multiples of √3.

We also have some triangle edges that are multiples of 2/3 √3. Clearly, the parity requirement applies to these edges too, that is unless they can be combined to yield integer multiples of √3.

Triangles T3 and T5

Triangles T3 and T5 are actually forced to combine into a single, larger triangle. But, before proving that, we need to state two principles concerning the position of triangle edges:

- For a right-angle triangle at least **one** edge must lie in the rectangle interior. (6)
- For a non-right-angle triangle at least **two** edges must lie in the rectangle interior. (7)

Applying (7) to triangle T5, we see that at least one 2/3 √3 edge lies in the rectangle interior. There is only one edge that can oppose this, and this is the 2/3 √3 edge of triangle T3. This leaves the remaining 2/3 √3 edge of T5 and the 4/3 √3 edge of T3. These two edges cannot oppose each other nor lie on the sides of the rectangle because of the parity principle. Therefore these two edges must be collinear, so that their total (2/3 √3 + 4/3 √3 = 2√3) can be opposed either by the rectangle or triangles edges that are multiples of √3. Triangles T3 and T5 can now be positioned relative to one another, as shown in Figure 2.

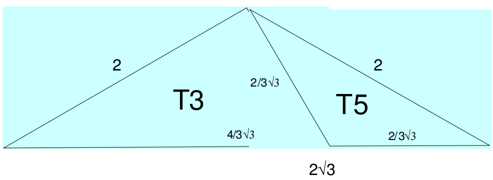

Figure 2.
Triangles T3 and T5 combined

Triangle	Edge 1	Edge 2	Edge 3	Right Angle
T1	2	2	2	No
T2	3	√3	√3	No
T4	√3	√3	√3	No
T6	1	2	√3	Yes
T3+5	2	2	2/3 √3	No

Table 3.
Triangle with T3+5 replacing T3 and T5

Since this composite triangle of T3 and T5 (call this T3+5) is forced, we can work with the five triangles listed in Table 3 instead of the six listed in Table 1. Note also that since we no longer have triangle edges that are multiples of 2/3√3, the case a = 3 in Table 2 is eliminated, leaving the two cases a = 2 and a = 4 to deal with.

Positioning the pieces

Table 3 now allows us to work out which triangle edges lie on the boundary of the rectangle and which lie in the interior. As we proceed, the interior edges will be **shaded**, the ones on the boundary will be outlined. The first few steps are common to the two cases a = 2 and a = 4.

Since T1 has three equal edges we can mark two out of the three as lying in the rectangle interior. The same reasoning applies to T4. Triangle T2 has two √3 edges, so at least one of them must lie in the rectangle interior. The same reasoning applies to the two length 2 edges of T3+5. The assignments are shown in Table 4.

Triangle	Edge 1	Edge 2	Edge 3	Right Angle
T1	2	**2**	**2**	No
T2	3	√3	**√3**	No
T4	√3	**√3**	**√3**	No
T6	1	2	√3	Yes
T3+5	2	**2**	2√3	No

Table 4. Initial assignment of triangle edges to interior and boundary

From here onwards, the solution procedure diverges for the cases *a* = 2 and *a* = 4.

Case *a* = 2

The case uses Table 4 as a starting point. There remains a total of 5√3 unassigned (non-integer) triangle edge. This forces the 2√3 of T3+5 to the rectangle boundary, which requires a total of 4√3. (To do otherwise would leave only a total of 3√3 for the boundary.) This boundary assignment also forces T3+5's remaining length 2 edge to the interior. The results so far are shown in Table 5.

Triangle	Edge 1	Edge 2	Edge 3	Right Angle
T1	2	**2**	**2**	No
T2	3	√3	**√3**	No
T4	√3	**√3**	**√3**	No
T6	1	2	√3	Yes
T3+5	**2**	**2**	**2√3**	No

Table 5. Second stage assignment of triangle edges when *a* = 2

Triangle	Edge 1	Edge 2	Edge 3	Right Angle
T1	**2**	**2**	**2**	No
T2	**3**	**√3**	**√3**	No
T4	**√3**	**√3**	**√3**	No
T6	**1**	**2**	**√3**	Yes
T3+5	**2**	**2**	**2√3**	No

Table 6. Third stage assignment of triangle edges when *a* = 2

At this stage the only unassigned integer edges are 1, 2, 2 and 3. With these edges, the only way to satisfy the two rectangle sides of length 2 is to use the two 2 edges of T1 and T6, which then requires the 1 and 3 edges to go to the interior. Since the length 2 edge of T6 is its hypotenuse, both its other edges must lie in the rectangle interior, so we can assign its remaining √3 edge to the interior. The remaining √3 edges in T2 and T4 must now be assigned to the rectangle boundary in order to complement the 2√3 edge of T3+5 and generate the total of 4√3 required by the rectangle. The results are shown in Table 6.

Case *a* = 4

The case uses Table 4 as a starting point. We can immediately assign the 2√3 edge of T3+5 to the rectangle interior, being longer than the rectangle's √3 dimension. We can also assign the remaining √3 edge of T2 to the rectangle interior because assigning it to the rectangle boundary would cause the triangle to lie outside of the rectangle due to the triangle's 120 degree angle. Now there remain only two √3 triangle edges to satisfy the √3 sides of the rectangle, so we can assign them both to the boundary. The results are shown in Table 7.

Triangle	Edge 1	Edge 2	Edge 3	Right Angle
T1	2	2	2	No
T2	3	√3	√3	No
T4	√3	√3	√3	No
T6	1	2	**√3**	Yes
T3+5	2	2	2√3	No

Table 7. Second stage assignment of triangle edges when *a* = 4

Triangle	Edge 1	Edge 2	Edge 3	Right Angle
T1	2	2	2	No
T2	3	√3	√3	No
T4	√3	√3	√3	No
T6	1	2	√3	Yes
T3+5	2	2	2√3	No

Table 8. Third stage assignment of triangle edges when *a* = 4

Now T6 is a right-angled triangle, and the assignment of its √3 edge to the rectangle boundary also forces its length 1 edge likewise and its length 2 edge to the interior.

At this stage only the length 1 edge of T6 is assigned to the rectangle boundary; but we are deficient by 7, and so all the remaining unassigned integer edges, which total 7, in T1, T2 and T3+5 can be assigned to the rectangle boundary. The results are shown in Table 8.

Assembling the puzzle

We now use Table 6 and Table 8 to assemble the puzzle pieces for cases $a = 2$ and $a = 4$ respectively. For the most part this process is forced if the pieces are chosen in the right order.

Case *a* = 2

First place the T3+5 triangle as shown along the $2\sqrt{3}$ rectangle side, as per Table 6 (see the left-hand diagram of Figure 3). Triangle T1 must occupy either the upper or lower edge of the rectangle, again as per Table 6. There appears to be a choice here, but it does not matter which way round T1 is placed, as will become clear shortly. For the moment, we will position it on the lower edge of the rectangle. Triangles T2 and T4 must now occupy the left-hand side of the rectangle, and T4 must take the upper $\sqrt{3}$ spot since it would overlap with T1 if placed in the lower spot. There is now only one place left for T6, and this completes the first solution. A second, distinct solution is obtained simply by “flipping” T3+5 (see the right-hand diagram in Figure 3). Finally, had we placed T1 at the upper edge of the rectangle, we would have obtained the same pair of solutions, except in mirror image form. This is easily verified by listing the triangles as they would appear clockwise or anticlockwise around the rectangle boundary. Thus we have just two distinct solutions for the case $a = 2$, ignoring rotations and reflections.

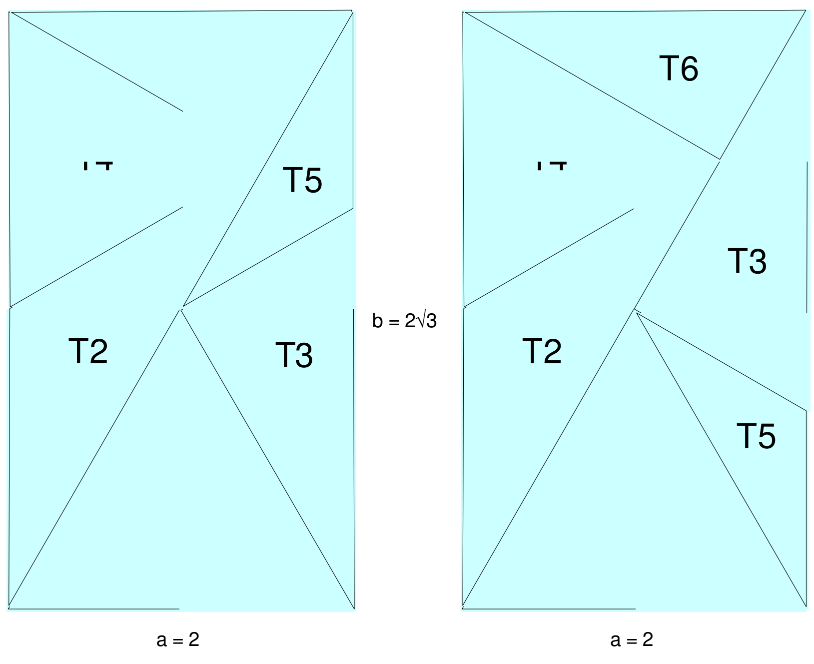

Figure 3. The two solutions for *a* = 2

Case *a* = 4

Triangles T4 and T6 are forced to the $\sqrt{3}$ ends of the rectangle as per Table 8 (see the upper diagram of Figure 4). Now the two integer edges of the rectangle are composed either of 2 + 2 or 1 + 3, and so we can place T2 as shown to complement the length 1 edge of T6. Then T1 must claim the right-hand half of the rectangle’s upper edge; otherwise it would overlap T4 and T2. Finally, T3+5 fits in the remaining space. As with case $a = 2$, T3+5 may be flipped to yield the other solution for case $a = 4$.

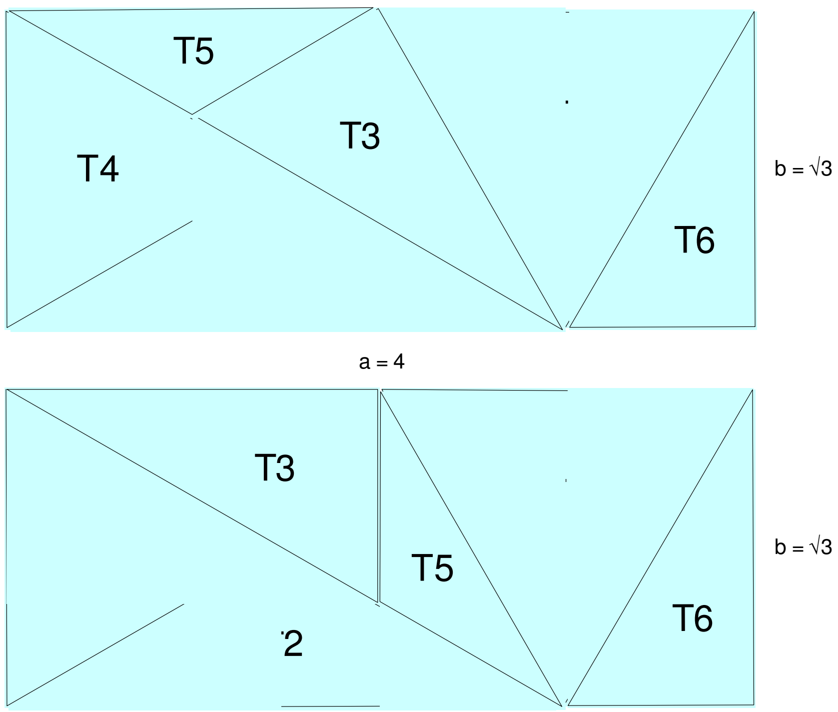

Figure 4. The two solutions for *a* = 4

Conclusions

The puzzle has been solved mainly through an edge matching technique, along with some properties of rational and irrational numbers. Surprisingly, purely geometric aspects played a minor part in all this.

Various other solution techniques were considered: one was to analyze the various angles in the rectangle and the six triangles; another was the possible use of Euler's theorem (V + F = E + 2) relating vertices, faces and edges in the solutions. Neither of these approaches helped though. There may be more elegant solutions to this puzzle, and the author would certainly be intrigued to hear of them.

A final thought: If triangle T1 is removed from Figure 4 and T6 is flipped and moved to the left, the rectangle shown in Figure 5 is produced with an area of 3√3. As before, T3+5 can be flipped to yield a companion solution (not shown). It is left as an exercise for the reader to prove that this is the only rectangle that can be formed with fewer than the six puzzle pieces.

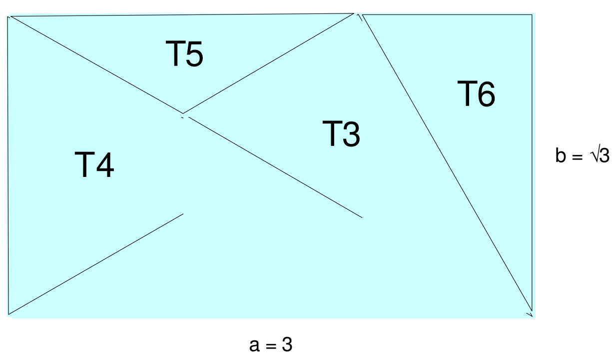

Figure 5. A rectangle using five puzzle pieces

Reference

[1] Ben Tanoff, *Langford's Legacy Continues,* CFF 81 (2010), pp 16-17.