# Puzzles with $3^1/_2$-ominoes

**by Eugène Neuzil and Suzanne Neuzil**

Torbijn [1] has drawn the interest towards $2^1/_2$-ominoes, that are the first mention of a new type of polyominoes for which we suggest the name of "fractional polyominoes". This domain of the polyominoes family [2, 3], to our best knowledge, has not hitherto been much investigated.

This communication is devoted to $3^1/_2$-ominoes, the upper homologues of $2^1/_2$-ominoes. They result from the association of three unit squares with a right-angled isosceles triangle ; the triangle has the area of a half-square and is obtained by cutting a square along its diagonal. The squares are linked along common edges ($s$) ; the triangle joins a square by one of its small sides ; in the different patterns described below, the hypotenuse ($h$) of the triangular part of a piece is connected to the hypotenuse of an adjacent item.

There are twelve basic $3^1/_2$-ominoes ("the animals"), represented in Figure 1, together with their mirror images ; apart from **1** and **O**, that are respectively irregular *convex* quadrilateral and pentagon, the other shapes are *concave* irregular polygons : pentagon (**2**), hexagons (**3**, **5**, **6**, **7**), heptagons (**4**, **8**, **10**) or octagons (**9**, **11**) ; the perimeter lenght has a general value of ($8s$ + $1h$), reduced to ($6s$ + $1h$) for **2** . The $3^1/_2$-ominoes derive from a tromino linked to an additional triangle : pieces **1**-**4** from the straight tromino, **5**-**11** and **O** from the right tromino. Piece **O**, the only symmetrical shape, is the only case where the triangle is linked to two unit squares ; the eleven other animals may thus be reversed (flipped over) to give their mirror images, indicated by an asterisk ; assigning the asterik to one of the two shapes present in a pair, for instance **2**-**2*** or **3**-**3***, is purely arbitrary. The rather small number of 12 basic $3^1/_2$-ominoes (not too few, not too many), the same number as for pentominoes and hexiamonds, appears particularly convenient to be proposed to a large audience, ranging from amateurs of puzzles to specialists of combinatorial geometry.

## Puzzles with 12 reversible $3^1/_2$-ominoes

The twelve $3^1/_2$-ominoes occupy an area of 42 unit squares ; they may be assembled to cover entirely a 6 x 7 rectangle. Some solutions are given in Figure 2. Solutions are generally easily found ; they are numerous, showing a great variety in the ratio of basic pieces *vs*. mirror images. The solutions indicated in the figures of this paper always show a lesser ratio of mirror images : this is obtained, when necessary, by reversing the entire pattern. In some patterns, the 6 x 7 rectangle results from the union of two adjacent smaller rectangles, of areas 2 x 7 and 4 x 7, as shown in Figures 2*a* and 2*b* (solution 2*b* encompasses no mirror images of the basic pieces). Another type of division gives two rectangles of identical size 3 x 7 (Figure 2*c*). In both cases, the relative position of the small rectangles, their rotation or reflection, lead to eight different configurations. On the opposite, a solution allowing no evident possibilities of modification is shown in Figure 2*d*.

The two rectangles 3 x 7 from a 6 x 7 solution may be linked to form a new 3 x 14 pattern, also leading to eight different configurations.

If cubes and half-cubes instead of squares and half squares are used, we obtain twelve solid planar $3^1/_2$-ominoes, corresponding to a volume of 42 unit cubes. These solid items can occupy a 2 x 3 x 7 box. Stacking two 3 x 7 patterns gives a simple solution, with no communication between the lower and upper layers. A solution in which 5 solid pieces participate to the two layers is depicted in Figure 3.

## Puzzles with 23 irreversible $3^1/_2$-ominoes

The 11 asymmetrical $3^1/_2$-ominoes, their 11 miror images (22 pieces that cannot be reversed) and the symmetrical piece **O** cover together a total area of 23 x $3^1/_2 = 80^1/_2$ square units, a number very close to 81; they all may be associated in a 9 x 9 square, leaving a triangular *hole* of one half square area. Among the different positions and possible orientations of the hole, we have selected two patterns : in Figure 4*a*, the hole touches the border of the 9 x 9 square, whereas in Figure 4*b* the hole is close to the center of the puzzle . When a solution is obtained, the position of the triangular hole may be characterized by superimposing the pattern on a 9 x 9 square containing a 5 x 5 grid (Figure 5*a*) ; the hole of the puzzle always coincides with one of the triangles of the grid, eventually after rotation and reflection. In the corner square of a 9 x 9 puzzle, the position of the hole given in Figure *5b* cannot obviously be retained.

Other puzzles using the twenty three $3^1/2$-ominoes may be also considered :

- A 3 x 27 rectangle, with the same area of 81, including also a half square triangle.
- The rectangles (4 x 20, 5 x 16 or 8 x 10), of an area of 80, lack a half square to receive the twenty three $3^1/2$-ominoes. These three rectangles must therefore be supplemented externally by a half square linked by one of its small sides to the perimeter of the rectangles. Excepted **0**, **7** and **7***, any one of the other shapes can provide the outer triangle which may be called a "thorn".
  Regarding the 8 x 10 rectangle, this thorn may occupy 8 + 10 = 18 positions on the perimeter, corresponding to 18 different "prickly" puzzles ; one of them is shown in Figure 6..

## Puzzles with fixed orientation $3^1/2$-ominoes

In most polyomino puzzles, the individual shapes may be reversed (flipped over) and rotated ; in some problems [4, 5], their rotation is not allowed and their orientation is fixed. Each piece must then be represented by *four* different *fixed orientation polyominoes* or *translation only polyominoes* (in some cases by only two, as for the domino, whereas rotation does not change the orientation of the X pentomino).

The orientation of the asymmetrical basic $3^1/2$-ominoes (and of their mirror images) is clearly indicated by the direction of their triangular part which may point at north (**N**), east (**E**), west (**W**) or south (**S**) ; the direction of the hypotenuse of the symmetrical **O** may be **WN, NE, ES** or **SW**. There are therefore 12 x 4 = 48 fixed orientation *reversible* $3^1/2$-ominoes, with a total area of 48 x $3^1/2$ = 168 unit squares ; these numbers increase up to 23 x 4 = 92 when fixed orientation *irreversible* $3^1/2$-ominoes are considered, and up to 322 for their total surface. The 92 different shapes may be named the *total* $3^1/2$-ominoes

*Puzzles with 48 fixed orientation reversible 3¹/2-ominoes*

The total surface of 168, covered by these 48 animals, is close to 169, the area of a 13 x 13 square. As for the design of Figure 4, a square is particularly interesting for studying tiling possibilities with fixed orientation polyominoes, on account of the presence of its $A_4$ rotation axis. Tiling this square with the 48 fixed orientation reversible 3¹/2-ominoes is a problem very easily solved, when starting from any one of the numerous solutions obtained for the 12 shapes of a 6 x 7 rectangular puzzle. A 6 x 7 solved puzzle is first submitted to three successive clockwise 90° rotations ; four patterns, α, β,γ and δ, oriented in directions **N, E, W** and **S,** are thus obtained :

$$\begin{matrix} \alpha & \beta \\ \delta & \gamma \end{matrix} \qquad \begin{matrix} \alpha & \delta \\ \beta & \gamma \end{matrix} \qquad \begin{matrix} \gamma & \beta \\ \delta & \alpha \end{matrix} \qquad \begin{matrix} \gamma & \delta \\ \beta & \alpha \end{matrix}$$

They are placed in the 13 x 13 square, respecting their orientation ; they leave an unique hole of area 1 at the centre of the square , as shown in Figure 7.

Several polygons circumscribe in area of 168 squares and may thus accept a complete tiling (without any hole) with the 48 fixed orientation reversible 3¹/2-ominoes. The **T** form octagon of Figure 8, for instance, can be filled with four identical copies, properly oriented, of a resolved 6 x 7 puzzle : as for the puzzle of Figure 7, four different solutions result from the translations of the four constitutive rectangles. Tiling the six rectangles 3 x 56, 4 x 42, 6 x 28, 7 x 24, 8 x 21 and 12 x 14 are presently investigated : these puzzles appear much more difficult to solve.

*Puzzles with 92 fixed orientation irreversible 3¹/2-ominoes*

As indicated above, the number of 23 fixed orientation irreversible 3¹/2-ominoes must be quadruplicated when the corresponding fixed orientation animals are considered. Such an important number supposes a probably very long search for tiling the rectangles 7 x 46 and 14 x 23, which both possess the requisite area of 322 squares.

Placing the 92 items in a 18 x18 square, of a surface of 324, gives a rapid solution, as a complete tiling is obtained with four identical copies of a 9 x 9 puzzle, conveniently oriented and supplemented by a hole of surface 2 or by holes totalizing the same area. To solve the 18 x18 puzzle, we have first selected a 9 x 9 puzzle ; a pattern with a triangular hole situated in the corner of the square (Figure 5*c)* is specially interesting, as the final result result is the aesthetic design depicted in Figure 9 : the four triangular holes of the constitutive smaller puzzles arer linked together to form a central unique square hole of area 2. Modifying the relative positions of the four small puzzles 9 x 9 by translations, results in changing the central square hole :

- to two triangles of surface 1 (one solution),
- to one triangle of surface 1 + two triangles of surface $^1/_2$ (two solutions),
- or to four triangles of surface $^1/_2$ (four solutions),

as shown in Figure 10.

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We wish that our few puzzles will inspire new research ; future results may help the $3^1/_2$-ominoes to leave the narrow lane where they are presently confined to gain a larger place in the vast field of Recreational Mathematics. Half a century separates the first description of the twelve patterns obtained by linking five squares (H. E. Dudeney, *Canterbury Puzzles*, 1907), from the paper of M. Gardner (*Scientific American*, May 1957), who brought the attention of a large public towards S. W. Golomb and his polyominoes. We hope to have not so long to wait !

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**References**

[1] P. J. Torbijn, *Problem 2621 : Symmetric Assemblies of 2½-ominoes*, Journal of Recreational Mathematics, 2002-03, **31**, p. 300. *Solutions :* Ray C. Ellis Jr,. Robert Wainwright and the Proposer, JRM , 2003-04, **32**, p. 342 and p. 344.

[2] Solomon W. Colomb, *Polyominoes*, Princeton University Press, New Jersey, 2nd edition, 1994.

[3] George E. Martin,, *Polyominoes. A Guide to Puzzles and Problems in Tiling,* The Mathematical Association of America, 1991.

[4] Dirk Wilms, *Problem 1943 : Total pentomino problems*, Journal of Recreational Mathematics, 1992, **24**, p. 58. *Solutions :* by the Proposer and Jacques Haubrich, JRM, 1993, **25**, pp. 75-76.

[5] Michael Reid, *Problem 2641 : Fixed Orientation Tetrominoes,* Journal of Recreational Mathematics, 2003-04, **32**, pp.166-167.

**Figure 1.** **The twelve $3\frac{1}{2}$-ominoes and their mirror images**
The triangular parts of the shapes are shaded.

**Figure 2.** **Four 6 x 7 solutions with twelve reversible $3\frac{1}{2}$-ominoes**
In all figures, asterisks indicate mirror images.

**Figure 3.** **Construction of a 2 x 3 x 7 solid using 12 solid $3\frac{1}{2}$-ominoes**

**Figure 4.** **Two 9 x 9 solutions with twenty three irreversible $3\frac{1}{2}$-ominoes**
In the *4b* solution, all the basic pieces are adjacent, and separated from their mirror images, are also linked together.

**Figure 5.** **Situation of the triangular hole in the 9 x 9 puzzle**

**Figure 6** **A solution with twenty three irreversible $3\frac{1}{2}$-ominoes in a 8 x 10 rectangle**

**Figure 7.** **A 13 x 13 puzzle with 48 fixed orientation reversible $3\frac{1}{2}$-ominoes**

**Figure 8.** **A T form octagon with 48 fixed orientation reversible $3\frac{1}{2}$-ominoes**

**Figure 9.** **A 18 x 18 puzzle with 92 fixed orientation irreversible $3\frac{1}{2}$-ominoes**

**Figure 10** **Different situations of the triangular corner hole of the 9 x9 rectangles forming the 18 x 18 pattern**
The area of the triangular hole has been considerably increased.

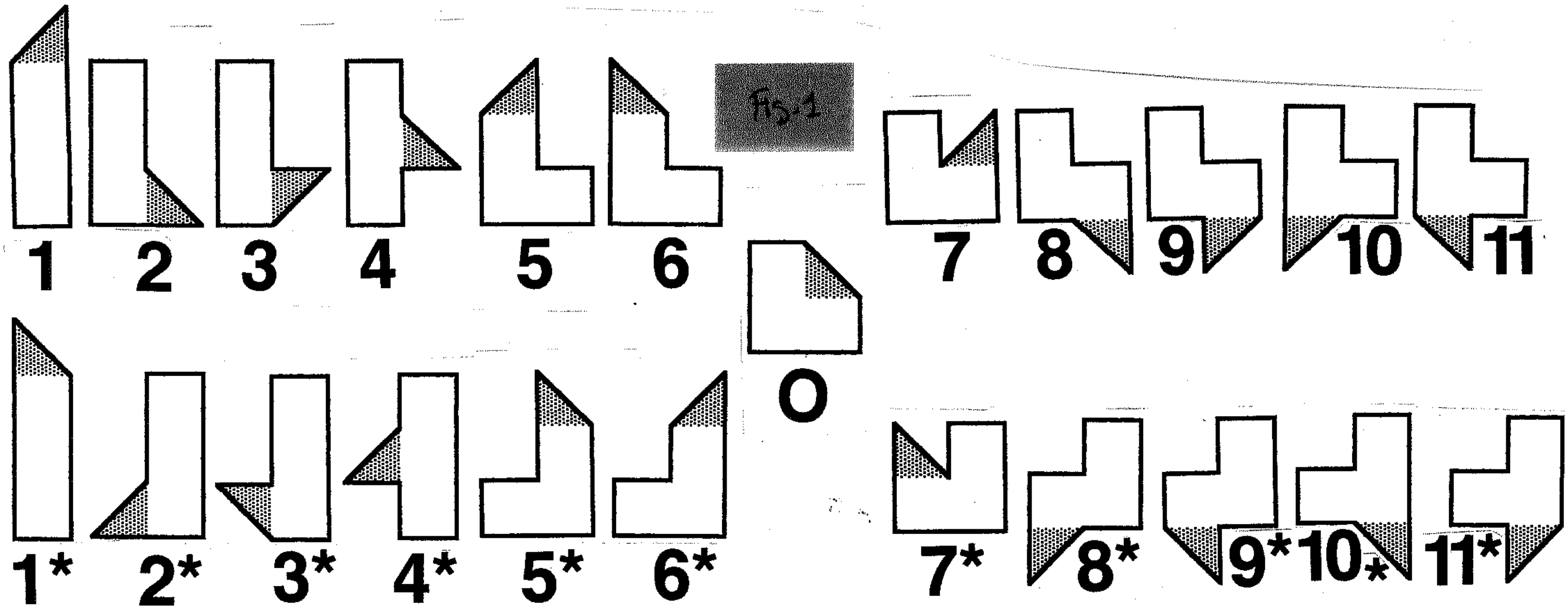
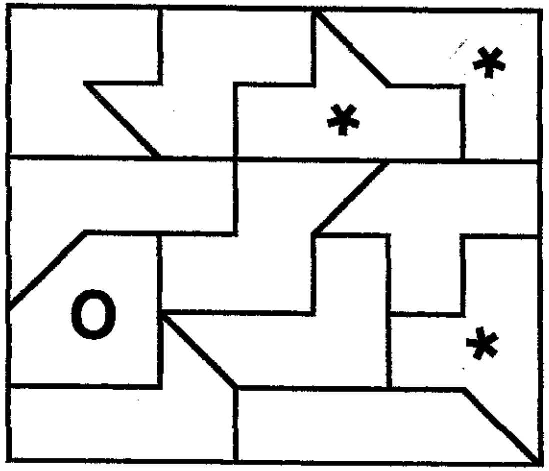
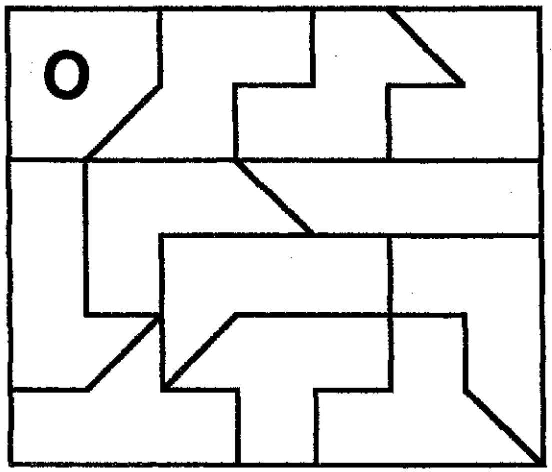


Fig. 1

Fig. 2



*a*



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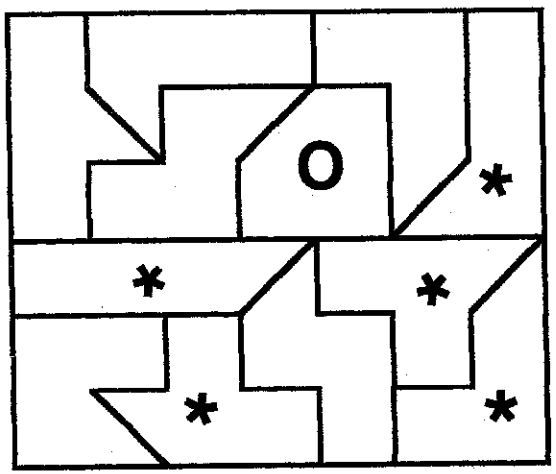

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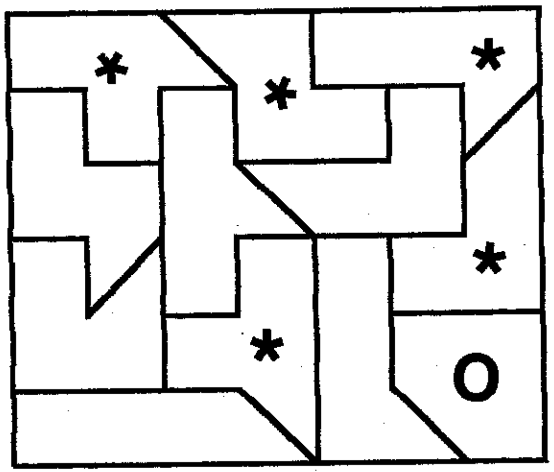
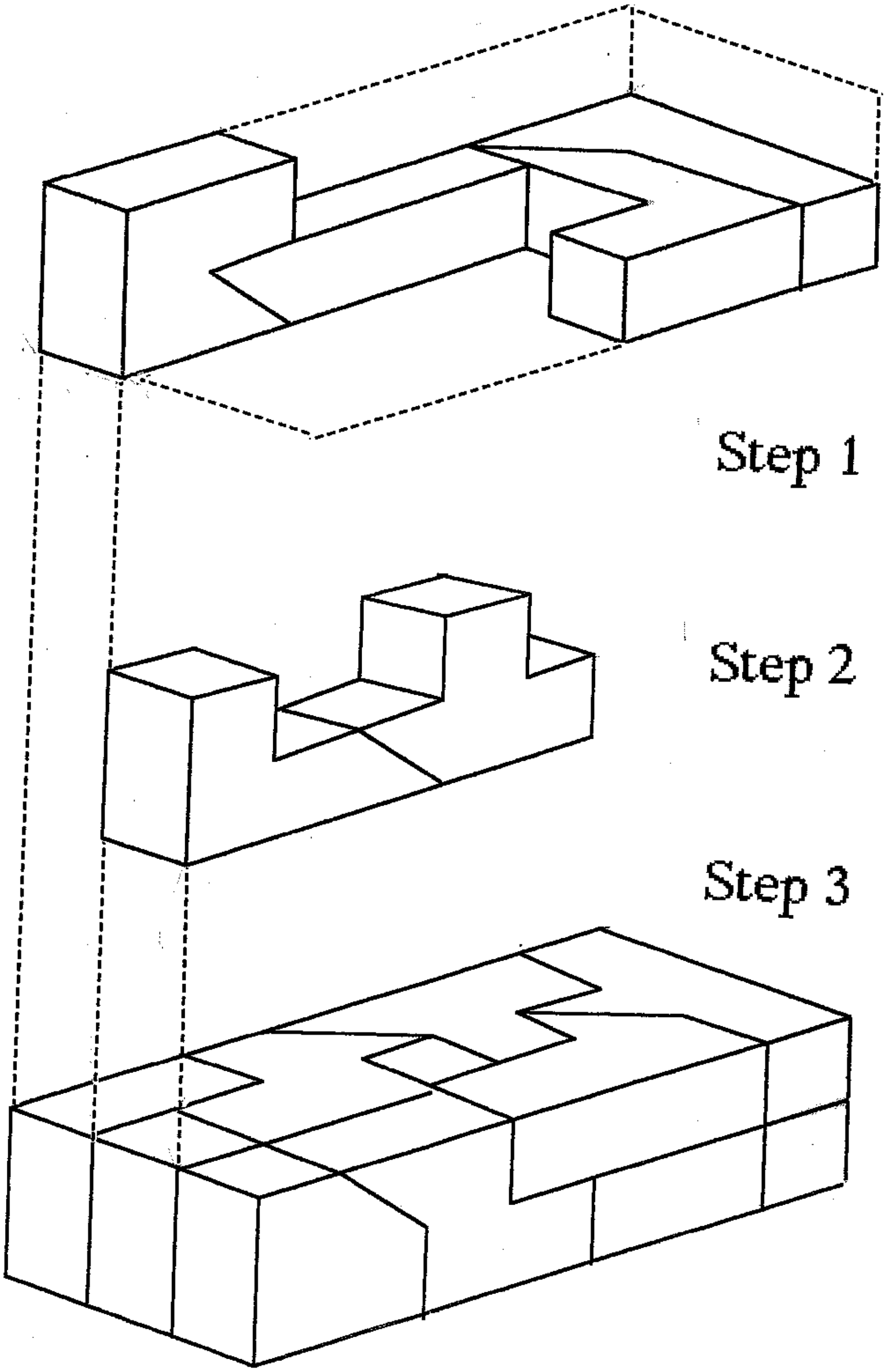
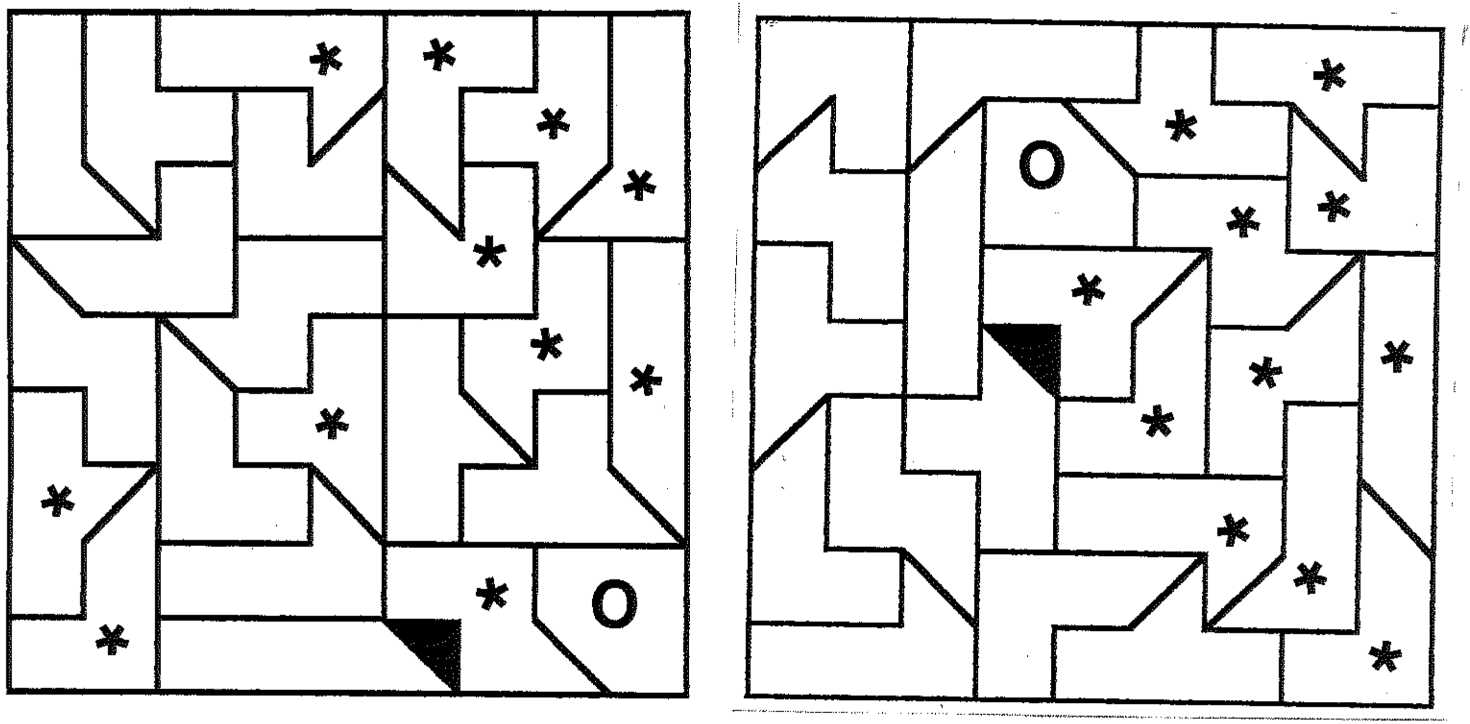


*d*

Fig-3





*a* *b*

Fig. 4

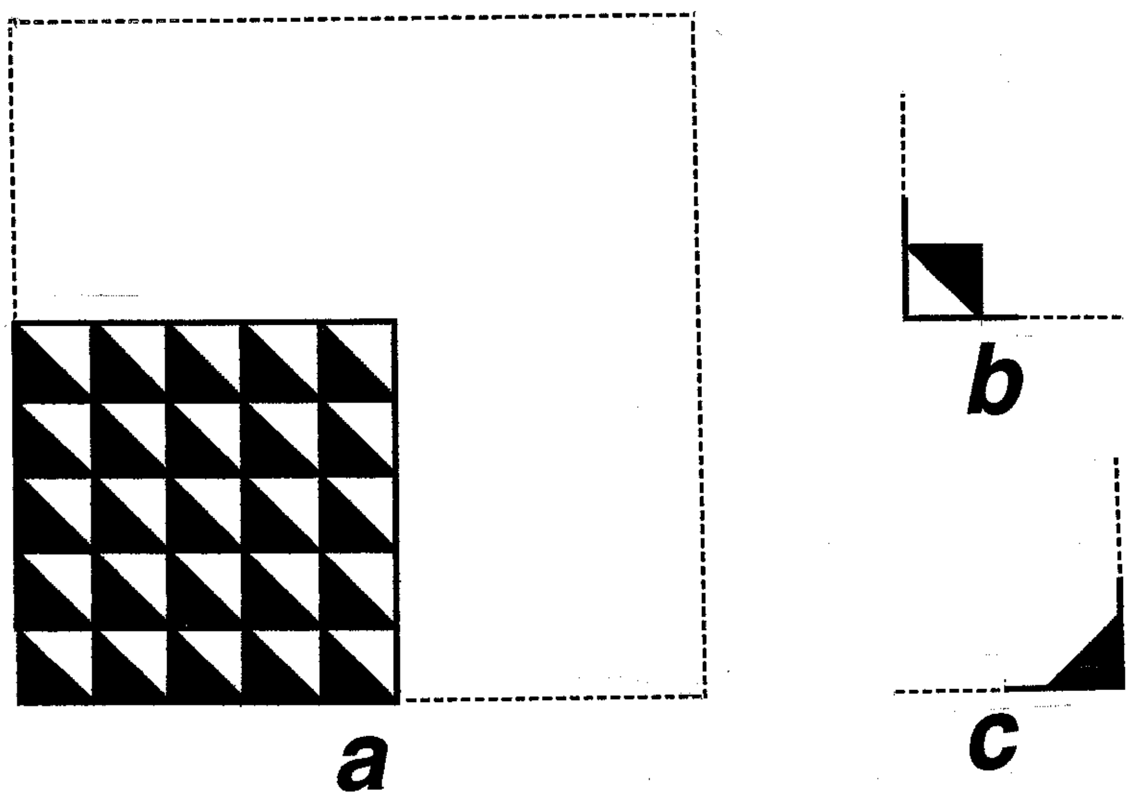



Fig 6

Fig 6

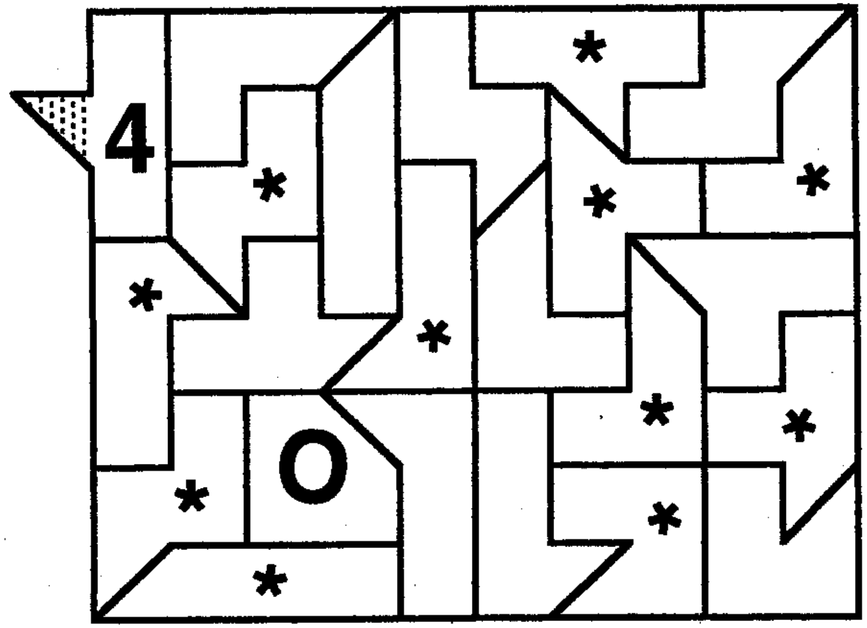
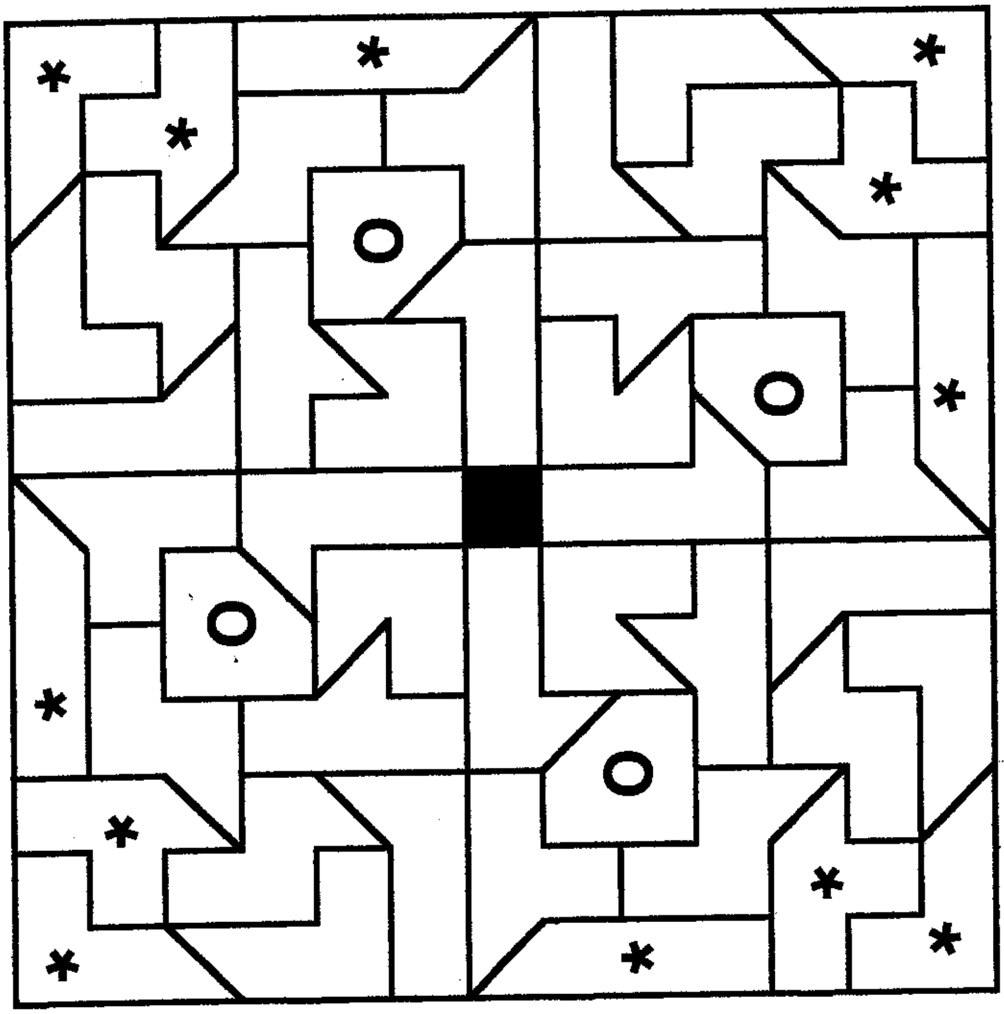



Fig 6

Fig. 7



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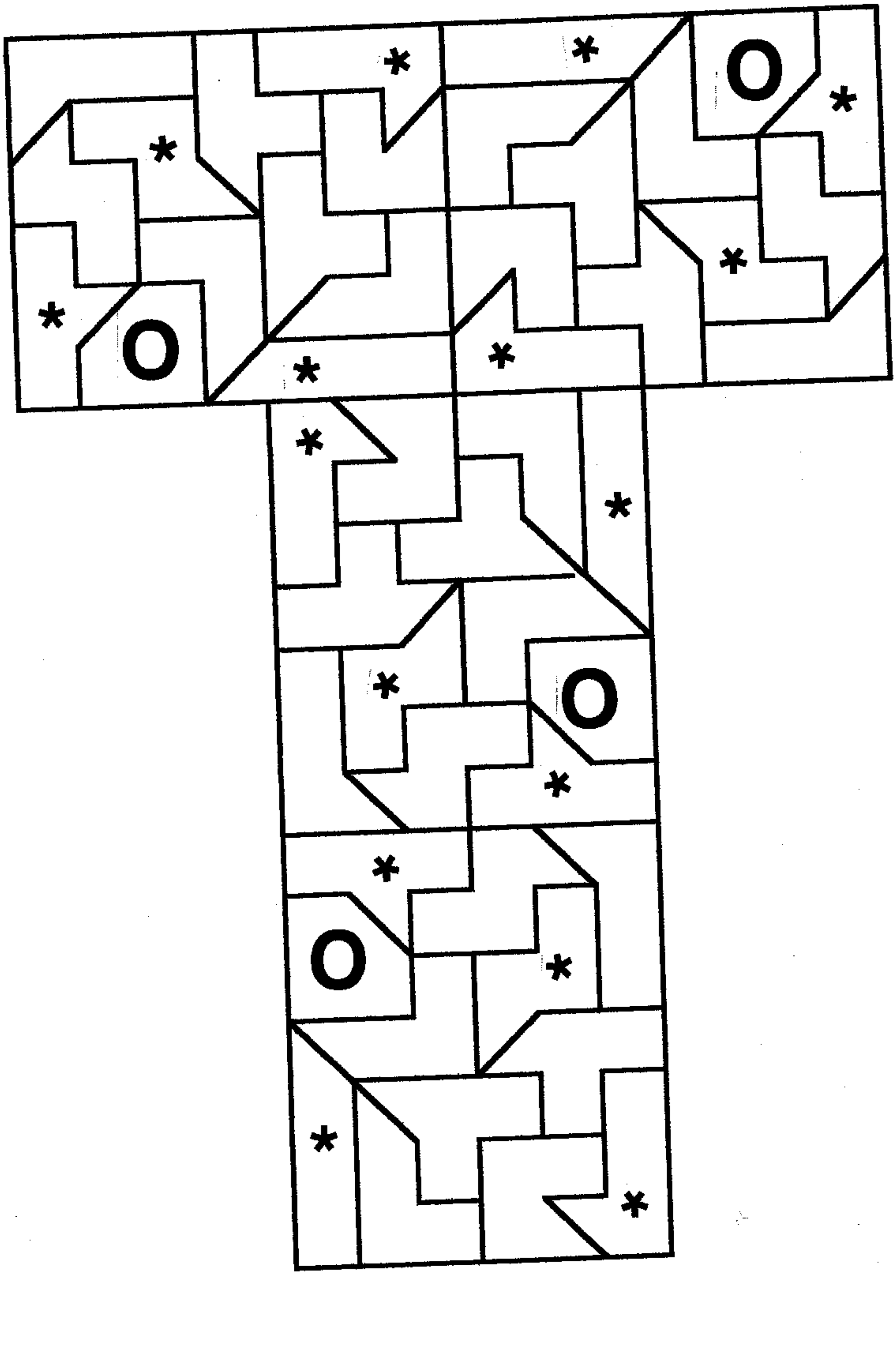
TOP

Fig 8



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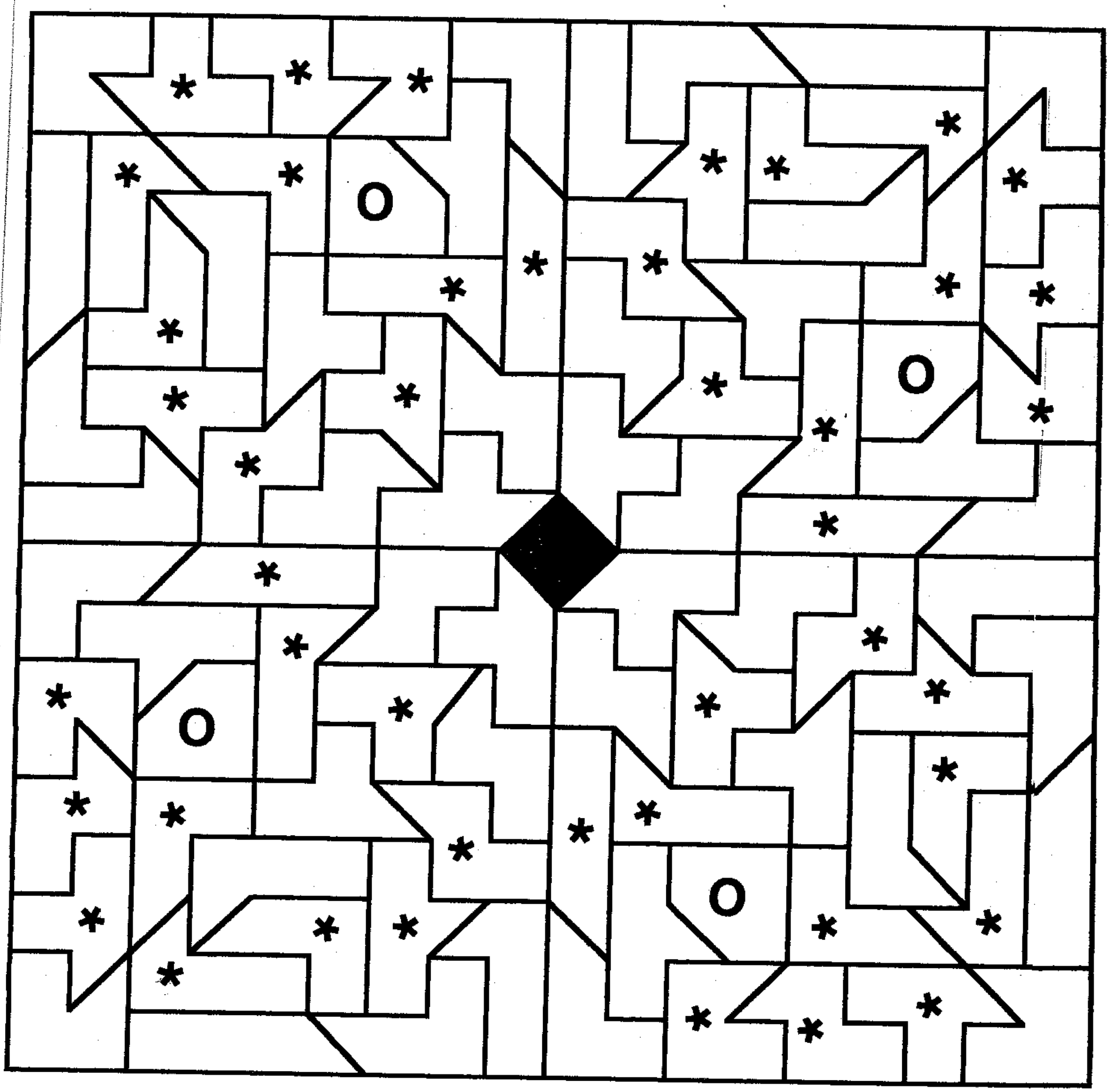

Fig. 9

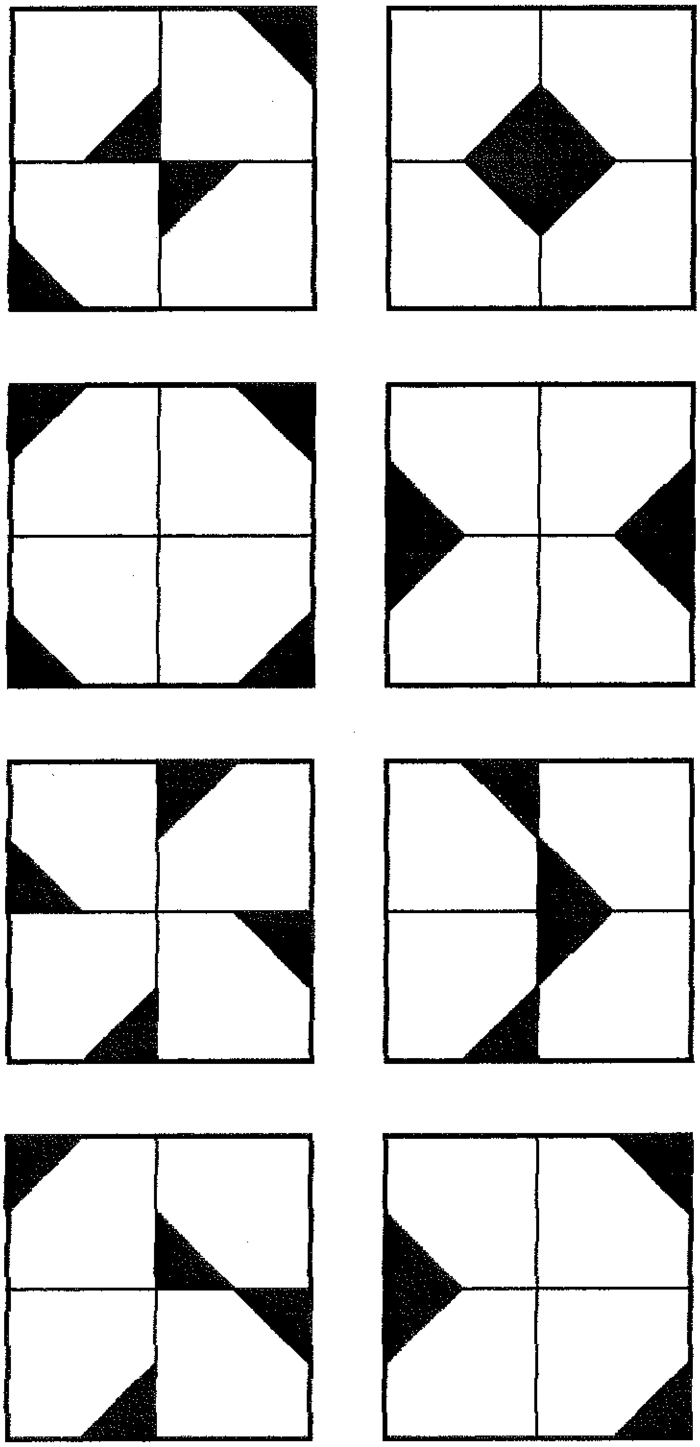

Fig. 10