Rubik-Sudoku
by Eduard Baumann

I have found on the market a Rubik's cube with the coloured stickers replaced by the numbers 1 to 9. Each cube side must not be uniformly coloured but must contain each number exactly once. This is an element of the Sudoku puzzles. But contrary to the Sudoku puzzles we have no relations between the different sides of the cube. What is the interest for this combination of ideas?

There is a supplementary aspect in such a Rubik-Sudoko. The numbered stickers are aligned on each side of the cube (for example all numbers upward on one side). You can manufacture millions of such Rubik-Sudoku by simply permuting the 9 stickers within a cube side, but not changing the direction of the numbers. Changing also the direction of the numbers of each cube side gives even more individual designs of Rubik-Sudokus.

Which particularities has the Rubik-Sudoku I bought which is shown below in Figure 1 and Figure 2. (front right top sides and back left bottom sides)?

Figure 1. The Rubik-Sudoku (front)

Figure 2. The Rubik-Sudoku (back)

(a) The directions of the sides of the cube

How many different arrangements of directions of the sides exist for a cube with 6 sides? If you fix one side (for example the front side with upward numbers) then you have 4 possibilities for each of the other 5 sides. This gives 4^5 = 1024 possibilities. If you are allowed to reorient the whole cube in each of these 1024 cases your are left with only 192 (see Lemma of Polya [1]) really different cases. Hence there exist 191 other direction designs than my special Rubik-Sudo.

(b) The permutation of the numbers in each side

The numbers in each side of my Rubik-Sudoku cube are chosen so that all Rubik cubicles are unique. Here is the list of the 12 edge cubicles: 12, 24, 25, 28, 33, 36, 37, 49, 58, 68, 79, 89. The list of 8 corner cubicles is the following: 159, 173, 173, 254, 285, 467, 496, 689. See also Figure 3. Two of these have the same 3 numbers 173. You can differentiate them in taking in account also the directions of the 3 numbers. In the left one 1 and 3 are parallel and in the other the 1 and 7 are parallel. We see that all the 20 cubicles are unique. They also have unique home places. I am sure that there is only one solution for my special Rubik-Sudoku.

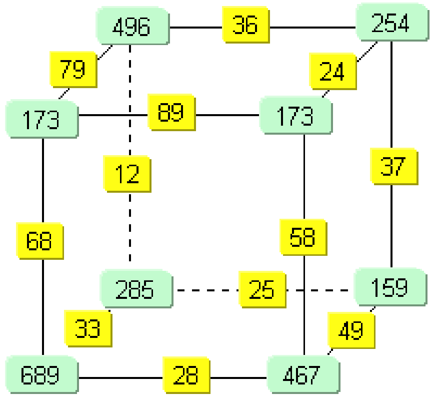

Figure 3. The arrangements of the numbers on the cube

You have to carefully note the positions of the unscrambled cube. Then you can solve the Rubiks cube with normal sequences. Without having noted the positions it is certainly extremely difficult to solve this Rubik-Sudoku.

Let us think about a modified form of my Rubik-Sudoku where all numbers are replaced by upward arrows. We still have almost unique cubicles! There exist exactly 8 different arrow arrangements for the 3 sticker corners. They are all present and therefore different. The arrow arrangements in the edge cubicles are not all unique. There is the group of the 4 elements 12, 37, 58 and 68 with parallel numbers side by side and the group of the 2 elements 25 and 36 with parallel numbers one on top of the other. Now we can calculate the multiplicity of the solution of this reduced (arrowed) Rubik-Sudoku puzzle. In the group of 4 identical edge cubicles only the 4!/2=12 even permutations are possible because the smallest edge cycle has 3 elements. To this 12 permutations we have to add 6 possibilities with an exchange in the 2 group together with an exchange in the 4 group. So we get a multiplicity of 18.

Reference

[1] Nick Baxter, *The Burnside DiLemma: Combinatorics and Puzzle Symmetry*, in Tribute to a Mathemagician, ISBN 1-56881-204-3, A.K.Peters, 2005, pp 199-210.